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(Mar. 13 2002)

We show that non-maximal entangled states can be used for implementing, with unit probability, remote generalized measurements (POVMs). We show how any n-qubit POVM can be applied remotely and derive its entanglement cost. The later turns out to be equal to the entanglement capability for a class of POVMs. This suggests a one-to-one relation between sub-sets of POVM operations and entanglement.

Although quantum entanglement has been a major research topic for the last decades, the nature of the relation between entanglement non-locality and the properties of physical interactions is a fairly new subject. During the recent years it was realized that entanglement can be used as a resource for implementing various types of remote interactions and operations [1-11]. Optimal ways for generating entanglement using a given interaction have been searched [12-14]. In particularly in ref. [5] an interesting qualitative general conjecture has been raised, while [7,8] pointed out a detailed connection between entanglement and operations. Nevertheless, the fundamental question mentioned above still seems to be open. In this letter we address this question.

Most known implementations of non-local operations, either require maximal-entangled states, or become probabilistic when non-maximal states are used. Nevertheless in exceptional cases non-local operations can be performed with unit probability and non-maximal entanglement [7,8].

The main purpose of this letter is to show that non-maximal entangled states can in-fact be used for implementing, with unit probability, a remote generalized measurement, usually referred to as a POVM (positive operator valued measure) [15]. We show that any n-qubit POVM can be measured remotely with certainty by using local operation and classical communication (LOCC), and single non-maximal entangled state. We also provide a general relation between an n-qbit POVM and the required amount of entanglement, which turns out to be equal to the entanglement capability of the POVM. We can hence classify the space of POVM operations into sub-sets, each having a definite entanglement measure. This suggests a one-to-one non-asymptotic relation between sub-sets of POVM operations and entanglement.

It will be helpful to begin with a concrete example. Suppose that one bit is encoded by two non-orthogonal states

$$|\psi_\pm\rangle = \alpha|0\rangle \pm \beta|1\rangle. \tag{1}$$

This bit cannot be retrieved back with certainty, however a generalized measurement, allows us to distinguish (sometimes) with certainty between $|\psi_\pm\rangle$.

Suppose we hand Bob the qubit, and informs *only* Alice (that is located remotely), what are the possible states $|\psi_\pm\rangle$. How can Alice and Bob measure the POVM? Surely, Bob can teleport [16] his state to Alice which then proceeds to perform the POVM. We show however that the POVM can be applied with optimal efficiency with less than one ebit of entanglement. (i.e. without teleportation).

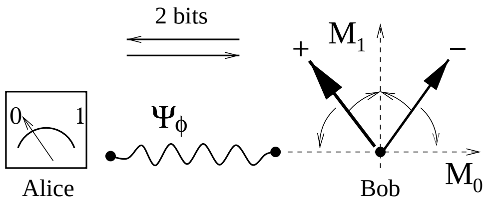

FIG. 1. Alice applies remotely either M_0 or M_1 on Bob's state. In the first case Bob can proceed to measure in the x-direction and distinguish with certainty between $\psi_\pm$. The operation uses a non-maximal entangled state Ψ_ϕ and one bit in each direction.

Let us consider first a POVM that allows to distinguish (sometimes) between $|\psi_\pm\rangle$ with certainty. A generalized measurement can always be described as a unitary operation acting on the system and an ancilla, followed by a projective measurement of the ancilla. In the present case we can use another qubit as the ancilla and operate on both the unitary U such that

$$U|0\rangle|\psi_\pm\rangle = |0\rangle M_0|\psi_\pm\rangle + |1\rangle M_1|\psi_\pm\rangle, \tag{2}$$

where

$$M_0 = \frac{1}{2}(1+\frac{\alpha}{\beta}) + \frac{1}{2}(1-\frac{\alpha}{\beta})\sigma_z,$$
$$M_1 = -\frac{\sqrt{\beta^2-\alpha^2}}{2\beta}(1-\sigma_z), \tag{3}$$

are non-unitary operators and $M_0^\dagger M_0 + M_1^\dagger M_1 = 1$. M_0 and M_1 are usually referred to as Kraus operators [17]. After measuring the ancilla, we find whether M_0 or M_1 have been generated. M_0 acts to rotate the non-orthogonal states $|\psi_\pm\rangle$ to $|0\rangle + |1\rangle$ or $|0\rangle - |1\rangle$, in which case we can distinguish with certainty between $|\psi_\pm\rangle$ by measuring σ_x of the system. If the ancilla is observed to be in the state $|1\rangle$, M_1 maps both states into $|1\rangle$, and the encoded information is completely lost. In this example the POVMs are $F_i = M_i^\dagger M_i$.

Let us now see how Alice can assist Bob remotely to perform this POVM (Fig. 1). While in the usual procedure the ancilla is at the hands of Bob, here we will have the ancilla with Alice. We start with a shared entangled state

$$\cos\phi|0\rangle_A|0\rangle_b + \sin\phi|1\rangle_A|1\rangle_b, \tag{4}$$

where the angle ϕ will be fixed in the sequel. Bob performs a controlled-NOT in the σ_z direction, with the qubit b as the control, and $|\psi_\pm\rangle_B$ the target. He next measures b in the σ_x direction, and sends one bit to inform Alice the outcome. Alice uses this information and by performing a π (or zero) rotation along the z-direction obtains the state

$$(\cos\phi|0\rangle_A + \sin\phi|1\rangle_A\sigma_z)|\psi_\pm\rangle_B, \tag{5}$$

where we ignored the state of b that factors out.

Next Alice applies a rotation that maps

$$\begin{aligned} |0\rangle_A &\to \cos\phi|0\rangle_A - \sin\phi|1\rangle_A, \\ |1\rangle_A &\to \sin\phi|0\rangle_A + \cos\phi|1\rangle_A, \end{aligned} \tag{6}$$

with the angle ϕ determined from:

$$\begin{aligned} \cos^2\phi &= \frac{1}{2}(1+\frac{\alpha}{\beta}), \\ \sin^2\phi &= \frac{1}{2}(1-\frac{\alpha}{\beta}). \end{aligned} \tag{7}$$

This transformation maps the entangled state of the ancilla and Bobs system to the desired form in the right hand side of equation (2). To completes the process Alice and Bob measure their systems and Alice sends one bit according to her result to Bob.

The main point is that by using only shared entanglement and LOCC, we reached an entangled state of same form as in the right hand side of eq. (2). The success probability, $\langle\psi_\pm|F_0|\psi_\pm\rangle$, is identical to that in an ordinary, local, POVM.

The entanglement consumed in the process

$$E_{POVM} = -\cos^2\phi\ln\cos^2\phi - \sin^2\phi\ln\sin^2\phi, \tag{8}$$

is generally less than one ebit. It goes to zero for $\langle\psi_+|\psi_-\rangle \to 0$, and tends to $E_{POVM} \to 1$ when $\langle\psi_+|\psi_-\rangle \to 1$. The classical communication cost is two bits one in each direction.

Clearly, in this example, the POVM dictates the amount of entanglement needed. Have we used instead a maximally entangled state, we could still generate M_0 remotely, but with a smaller probability of success. We can still make use of a maximally entangled state. Alice first dilutes to a non-maximal state, which she can do with unit probability [18], and then applies the above procedure. In passing we remark that the one bit sent from Bob to Alice is random. On the other hand the bit sent from Alice to Bob is biased according to the success probability of the POVM. Hence over many trails Bob can gain information on the inner product $\langle\psi_+|\psi_-\rangle$.

We next consider the problem in more general terms. Every POVM on system B may be realized by letting B first interact with an ancillary system A in a standard initial state, and then observe A.

$$U_{AB}|0\rangle_A|\psi\rangle_B = \sum_\mu |\mu\rangle_A M_\mu|\psi\rangle_B, \tag{9}$$

where the Kraus operators, $M_\mu = {}_A\langle\mu|U_{AB}|0\rangle_A$, satisfy $\sum M_\mu^\dagger M_\mu = 1$. The corresponding POVM, $F_\mu = M_\mu^\dagger M_\mu$, appears with the probability distribution $\text{Prob}(\mu) = {}_B\langle\psi|F_\mu|\psi\rangle_B$.

The measurement of the ancilla A realizes a particular (non-unitary) transformation on the system. On the other hand if we measure the ancilla in a different basis, or equivalently first apply a unitary U_A we obtain

$$U_A\sum_\mu |\mu\rangle_A M_\mu = \sum_{\eta,\mu}|\eta\rangle_A U_{\eta\mu}M_\mu = \sum_\mu |\mu\rangle_A N_\mu, \tag{10}$$

where

$$N_\mu = \sum_\rho U_{\mu\rho}M_\rho. \tag{11}$$

Therefore, a unitary transformation on the ancilla gives rise to a new set of Kraus operators.

Before we proceed it is instructive to compare our problem with the superoperator picture. If we do not observe the ancilla, the effect of U_{AB} on the subsystem B is described by a superoperator $\$_B\rho_B = \sum M_\mu\rho_B M_\mu^\dagger$. Two unitary related sets, such as M_μ and N_μ above, then represent the same $\$_B$. Nevertheless, when we do observe the ancilla, as in our case, we learn which Kraus operator has been realized on the system. Hence, two unitarily related Kraus sets generally give rise to inequivalent POVMs.

The realization of the POVM requires an interaction U_{AB} between the ancilla and the system. Our main goel is to find how to construct this transformation, using entanglement and LOCC, when the ancilla is located remotely with Alice. To this end we start with some preliminary steps.

Definition 1.: A a set of Kraus operators will be defined as an orthogonal set if

$$(M_\mu, M_\eta) \equiv \frac{1}{N_B}\text{Trace}M_\mu^\dagger M_\eta = c_\mu\delta_{\mu\eta} \tag{12}$$

where N_B is the dimension of Bob's system. For example, $M_\mu = \alpha_\mu\sigma_\mu$, $\mu = 0,...3$, where σ_k $k = 1,2,3$ are Pauli matrices and $\sigma_0 = 1$, constitute an orthogonal set.

Notice that the unitary transformation (11) generally *does not* preserve the inner produce (12). Clearly, if $N_\eta = \sum_\mu U_{\eta\mu}\alpha_\mu\sigma_\mu$, $\text{Trace}(N_\mu^\dagger N_\eta) \neq c_\mu\delta_{\mu\eta}$, unless all α_μ are equal. Therefore, in certain cases, by applying the unitary transformation (11) we may obtain from a non-orthogonal an orthogonal one.

Definition 2.: A set of Kraus operators M_μ will be said to admit an orthogonal equivalent set, or shortly OE, if it is unitarily related to an orthogonal set.

Of importance to us are OE sets that are up to a multiplicative constant proportional to a unitary. As we readily show, such orthogonal sets can be generated by local operations and entanglement. To see this, let us consider for simplicity a one-qubit POVM. Suppose Alice wishes to apply U_{AB} that leads to the orthogonal Kraus set $M_\mu = \alpha_\mu \sigma_\mu$. To do that, Alice and Bob need the entangled state

$$|\Psi\rangle_{Ab} = \sum_\mu \alpha_\mu |\mu\rangle_A |\mu\rangle_b. \tag{13}$$

Bob starts by performing a local unitary transformation between his system and his part (b) of the entangled state

$$U_{bB} = \sum_\mu |\mu\rangle_{bb}\langle\mu|\sigma_\mu, \tag{14}$$

and measures b in a complementary basis $|\eta\rangle_b$ with equal probability to get η. This leads to

$$\sum_\mu \pm\alpha_\mu |\mu\rangle_A \sigma_\mu, \tag{15}$$

where the $\pm$ signs in front of each term is determined according to Bob's outcomes for η_b. Alice then can correct them all to be $+$, according to the 2-bit message she received from Bob, by applying an appropriate rotation on the ancilla. Now we recall that a unitary acting on A induces a unitary acting on M_μ. Therefore, any Kraus set that is OE to the above orthogonal M_μ can be generated by Alice by means of an appropriate local unitary followed by a measurement which records to result of the POVM.

More generally, in order to find what are the POVMs that can be applied remotely, we need to check which Kraus sets are OE.

Theorem: *Any n-qubits POVM can be represented by an OE Kraus operator set.*

Let us start with a one qubit POVM. Since σ_μ forms a basis, we can expand $M_\mu = \sum c_{\mu\eta}\sigma_\eta$, where $c_{\mu\eta} = (\sigma_\eta, M_\mu)$. **Lemma:** M_μ is OE iff $c^\dagger c$ is diagonal. Proof: If $c^\dagger c$ is diagonal, the columns of the matrix c are orthogonal vectors. Hence c may be expressed as a product of a unitary and a diagonal matrix: $c_{\mu\nu} = \sum_\eta U_{\mu\eta}\delta_{\eta\nu}\alpha_\nu$. The reverse direction of the lemma is immediate.

Consider the conditions on the matrix c. From $\sum M_\mu^\dagger M_\mu = 1$ we obtain $\sum_{\mu\eta} |c_{\mu\eta}|^2 = 1$ and

$$\Re \sum_\mu c^*_{\mu 0} c_{\mu k} = 0, \tag{16}$$

$$\Im \sum_\mu c^*_{\mu n} c_{\mu m} = 0, \tag{17}$$

where $\Re$ and $\Im$ denote the real and imaginary parts respectively, and roman letters only run over $1, 2, 3$. These conditions are not enough to force every M_μ to be OE. A general Kraus sum representation does not admit a unitary equivalent orthogonal representation. (e.g. $M_0 \propto |\uparrow_z\rangle\langle\uparrow_x|$, $M_1 \propto |\uparrow_x\rangle\langle\downarrow_x|$).

Consider however a general one-qubit POVM F_μ. Since F_μ are a semi-positive hermitian operators, they can be described by (at most) four hermitian Kraus operators $M_\mu = \sqrt{F_\mu}$ and consequently the matrix c is real. Eq. (16) than implies that the 0'th and k'th columns of c are orthogonal. Next suppose that after applying U_{AB} in (9), Bob applies a local σ_1 rotation. Hence $U_{AB} \to \sigma_1 U_{AB}$. This induces another Kraus set obtained by $c_{\mu 0} \to c_{\mu 1}$, $c_{\mu 1} \to c_{\mu 0}$, $c_{\mu 2} \to i c_{\mu 3}$, $c_{\mu 3} \to -i c_{\mu 2}$. But now, because columns 2 and 3 are purely imaginary, we deduce from eq. (17) that the 1'st column must be orthogonal to the second and third columns. Similarly we obtain that all columns are orthogonal. Hence $\sqrt{F_\mu}$ is OE. It is straightforward to generalize the above considerations to an n-qubit POVM. This then concludes the proof.

As a corollary we conclude that: *Any n-qubit POVM can by generated remotely by the present method.*

Next let us quantify the entanglement resources needed to apply a POVM. The coefficients α_μ fix the schmidt coefficients of the needed shared entangled state. This is readily found be noticing that $c^\dagger c$ given by a diagonal matrix of the form $\alpha_\mu^2 \delta_{\mu\lambda}$. Therefore,

$$(\alpha_\lambda)^2 = \sum_\mu (\sigma_\lambda, \sqrt{F_\mu})^2. \tag{18}$$

The above expression can be extended to n-qubit POVM by replacing σ_μ with the basis $\sigma^1_\mu \sigma^2_\lambda \cdots \sigma^n_\eta$. The entanglement consumed for generating all POVMs which are unitary related to $\sqrt{F_\mu}$ is therefore

$$E_{POVM} = -\sum_\lambda \alpha_\lambda^2 \ln \alpha_\lambda^2. \tag{19}$$

The classical communication cost is determined by the number n of qubits on which we apply the POVM. It is at most given by n bits.

We remark that in this approach Alice has full control on the POVM and obtains the result of the measurement. For other purposes it may be useful to "share" between the job of performing the POVM between Alice and Bob. The first example we gave (Fig. 1) is indeed of that type. The entanglement needed for applying a "shared" POVM is obviously smaller.

To summarize: we showed that any n-qubit POVM (that may be also viewed as a set of generally non-unitary operations) can be implemented remotely. To this end, for each class of unitary related OE POVMs Alice and Bob need a particular entangled state which is determined by the POVM. For the special case of a remote projective (von-Neumann) measurement of n qubits we have $E_{POVM} = n$. After coupling the entangled particle to his system Bob performs a measurement and transmits the result to Alice. She transforms this state to the standard form of an orthogonal POVM. To apply the POVM she applies the unitary

$$U_{\mu\nu} = \sum_\eta (\sigma_\eta, M_\nu)\delta_{\eta\mu}\frac{1}{\alpha_\eta} \tag{20}$$

on her entangled ancilla and finally measures the ancilla. The efficiency of the process is optimal: i.e. the information gained is identical to a locally performed POVM.

Is the measure given in (18,19) unique?. Does it determine the minimal entanglement needed to perform the POVM? Let us show that if the action of the POVM on the system is given as in our case, by the semi-positive hermitian root $\sqrt{F_\mu}$, that is indeed true.

Consider the entanglement capability of the remote POVM defined by $E_{capability} = \max_{\psi_B} E(\Psi_{AB})$. I.e. we maximize the entanglement generated by the POVM between the ancilla of Alice and Bob's system. Next, let E_{cost} be the minimal entanglement needed to generate remotely the POVM. By the principle of entanglement non-increase under LOCC, we must have

$$E_{capability} \leq E_{cost}. \tag{21}$$

We have seen that the POVM can be performed using the entanglement E_{POVM} defined in (19). Since our method may not be optimal it can be that $E_{cost} \leq E_{POVM}$. Now consider the entanglement capability. Since our POVM is OE it can be transformed locally to the form $\sum |\mu\rangle_A \alpha_\mu \sigma_\mu |\psi\rangle_B$. Suppose that Bob's particle entangled with another local particle in an EPR state. In this special case the entanglement capability of the POVM is precisely E_{POVM}. Therefore we arrive to the inequality

$$E_{capability} \geq E_{cost}, \tag{22}$$

which when combined with (21) leads to the desired conclusion

$$E_{cost} = E_{capability} = E_{POVM}. \tag{23}$$

Our POVM construction therefore leads to a unique one-to-one relations between POVMs and entanglement. The entanglement E_{POVM} constitutes a lower bound on the entanglement cost [19] and an upper bound on the entanglement capability.

I would like to thank Berry Groisman and Sandu Popescu for very helpful discussions. The research was supported in part by grant 62/01-1 of the Israel Science Foundation, established by the Israel Academy of Sciences and Humanities and by the Israel MOD Research and Technology Unit.

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[19] A POVM does not uniquely defined the back-action on the observed system. When the back-action can be described by a hermitian $\sqrt{F_\mu}$, up to additional local actions of Bob, our result is valide. However consider unitaries performed remotetly by Alice before the POVM. This requires additional entanglement. It was not shown that in this case, the totat consumed entanglement is aways larger than E_{POVM}. I thank S. Popescu for this remark.